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Extremity of b-bistochastic Quadratic Stochastic Operators on 2D Simplex

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ABSTRACT

In the present paper we study the set of b-bistochastic quadratic stochastic operators. Using the description of such kind of operators, we investigate extreme points of the set of b-bistochastic quadratic stochastic operators (q.s.o) acting on two dimensional simplex. Further, we introduce weaker conditions of extreme points called *quasi-extreme* of the set b-bistochastic quadratic stochastic operators (q.s.o) and study its relations with extreme points of the set b-bistochastic linear stochastic operators defined on two dimensional simplex.

Keywords: Quadratic stochastic operators (q.s.o), b-order, bistochastic, doubly stochastic, b-bistochastic, extreme points.

1. Introduction

In the previous century, the theory of linear operators has been vigorously studied and it has been well developed ; so it is natural to consider nonlinear operators. The simplest case starts with a quadratic one. It turns out that quadratic operators have tremendous applications in various field such as physics, disease dynamics,evolutionary biology, economic and social systems (see Hofbauer and Sigmund (1998), Lyubich et al. (1992)). No matter the efforts of countless researchers that have introduced several different classes of quadratic operators, it is noteworthy to acknowledge the fact that it is not an easy task and therefore, the topic has still not been covered. Consequently, in this paper, we are focusing on quadratic stochastic operators which can be traced back to Bernstein's work Bernstein (1924) (see Mukhamedov and Ganikhodjaev (2015) for review) that he usd to describe the distribution evolution of individuals in a population. In addition, these operators were used as a crucial source of analysis especially in the study of dynamical properties and modelings in many different fields such as biology (Goel et al. (1971), Hofbauer and Sigmund (1998)), physics (Plank and Losert (1995), Takeuchi (1996)), economics and mathematics (Goel et al. (1971), Lyubich et al. (1992), Takeuchi (1996)).

One can comprehend the time evolution of species in biology by the following situation. By letting $I = 1, 2, \dots, n$ be the n type of species (or traits) in a population, we denote $x^{(0)} = (x_1^{(0)}, ..., x_n^{(0)})$ be a probability distribution of the species in early state of that population and the probability individual in the i^{th} species and j^{th} species cross-fertilize to produce an individual from k^{th} species (trait) be $P_{ij,k}$. Given $x^{(0)} = (x_1^{(0)}, ..., x_n^{(0)})$, we can find probability distribution of the first generation, $x^{(1)} = (x_1^{(1)}, ..., x_n^{(1)})$ by using a total probability, i.e.,

$$x_k^{(1)} = \sum_{i,j=1}^{n} P_{ij,k} x_i^{(0)} x_j^{(0)}$$

This relation defines an operator which is denoted by V and it is called *quadratic stochastic operator (q.s.o.)*. In other words, each q.s.o. describes the sequence of generations in terms of probability distributions $P_{ij,k}$. In Ganikhodzhaev et al. (2011), Mukhamedov and Ganikhodjaev (2015),they have given self-contained exposition of the recent achievements and open problems in the theory of the q.s.o. One of the main problems in this theory is to study the behavior of such kind of operators in which the difficulty of the problem depends on the cubic matrix $(P_{ijk})_{i,j,k=1}^{m}$. An asymptotic behavior of the q.s.o. even in the small dimensional simplex is complicated (see for example Zakharevich (1978)).

The majorization (see Hardy et al. (1986)) has an important impact on many branches of sciences. For example, the idea of majorization continues to be used in other fields such as chemistry and physics, but they used different names such as "x is more *mixed* than y", "x is more *chaotic* than y" and "x is more *disordered* than y". One of the examples is given by Zylka and Vojta (1991).

A new order called majorization introduced by Parker and Ram (1996). This new order opened a path for the study to generalize the theory of majorization of Hardy, Littlewood and Polya used in Hardy et al. (1986). Due to the fact that new majorization is defined as a partial order, it has additional applications compared to classical majorization (see also Helman et al. (1993)). Moreover, in Parker and Ram (1996), linear stochastic operators preserving the order are described which in the first place, sparked our interest to investigate the case for quadratic operators.

To differentiate between the terms majorization and classical majorization (Hardy et al. (1986)), from now onwards, we will consider majorization as b-order (which is denoted as $\leq^b$) while the classical majorization will be majorization (which is denoted as $\prec$). In Mukhamedov and Embong (2015), the definition of b-order preserving q.s.o., i.e. $V(\mathbf{x}) \leq^b \mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$, called *b-bistochastic* was introduced. Therefore, we have explored descriptive properties of b-bistochastic q.s.o. in Mukhamedov and Embong that allowed us to find sufficient conditions on cubic stochastic matrix to be a b-bistochastic q.s.o.(see Mukhamedov and Embong). To broaden our research, other properties of b-bistochastic q.s.o. have been analysed in Mukhamedov and Embong,Mukhamedov and Embong (2016). We point out that a q.s.o. which preserves the majorization (i.e. V is bistochastic, if $V(x) \prec x$ for all $x \in S^{n-1}$) was investigated in Ganikhodzhaev et al. (2012), Ganikhodzhaev (1993). In general, a description of such kind of operators is still an open problem.

In this paper, we continue our previous investigations on b-bistochastic operators. In section 2, we recall necessary results to describe extreme points of the set of b-bistochastic q.s.o. The main goal of this paper is to describe extreme points of the set of b-bistochastic q.s.o. on low dimensional simplices (see section 3). Lastly, in section 4, we introduce weaker conditions of extreme points called *quasi-extreme* of the set b-bistochastic q.s.o, and study its relations with extreme points of the set b-bistochastic linear stochastic operators.

2. b-order and b-bistochastic operators

Throughout this paper we consider the simplex

$$S^{n-1} = \left\{ \mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \quad \middle| x_i \geq 0, \quad \sum_{i=1}^{n} x_i = 1 \right\}. \tag{2.1}$$

Define the functionals $\mathcal{U}_k : \mathbb{R}^n \to \mathbb{R}$ $(k = 1, 2, \ldots, n-1)$ by

$$\mathcal{U}_k(x_1, \ldots, x_n) = \sum_{i=1}^{k} x_i. \tag{2.2}$$

Let $\mathbf{x}, \mathbf{y} \in S^{n-1}$. We say that $\mathbf{x}$ *b-ordered* or *b-majorized* by $\mathbf{y}$ ($\mathbf{x} \leq^b \mathbf{y}$) if and only if $\mathcal{U}_k(\mathbf{x}) \leq \mathcal{U}_k(\mathbf{y})$, for all $k = 1, \ldots, n-1$.

The introduced relation is a partial order i.e. it satisfies the following conditions:

(i) For any $\mathbf{x} \in S^{n-1}$ one has $\mathbf{x} \leq^b \mathbf{x}$,

(ii) If $\mathbf{x} \leq^b \mathbf{y}$ and $\mathbf{y} \leq^b \mathbf{x}$ then $\mathbf{x} = \mathbf{y}$,

(iii) If $\mathbf{x} \leq^b \mathbf{y}$, and $\mathbf{y} \leq^b \mathbf{z}$ then $\mathbf{x} \leq^b \mathbf{z}$.

where $\mathbf{x}, \mathbf{y}, \mathbf{z} \in S^{n-1}$. Moreover, it has the following properties:

(i) $\mathbf{x} \leq^b \mathbf{y}$ if and only if $\lambda\mathbf{x} \leq^b \lambda\mathbf{y}$ for any $\lambda > 0$.

(ii) If $\mathbf{x} \leq^b \mathbf{y}$ and $\lambda \leq \mu$, then $\lambda\mathbf{x} \leq^b \mu\mathbf{y}$.

Using the defined order, one can define the classical majorization (Marshall et al. (1979)). First recall that, for any $\mathbf{x} = (x_1, x_2, \ldots x_n) \in S^{n-1}$, we define $\mathbf{x}_{[\downarrow]} = (x_{[1]}, x_{[2]}, \ldots x_{[n]})$, where

$$x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$$

is nonincreasing rearrangement of $\mathbf{x}$. The point $\mathbf{x}_{[\downarrow]}$ is called *rearrangement* of $\mathbf{x}$ by nonincreasing order. Take two elements $\mathbf{x}$ and $\mathbf{y}$ in S^{n-1}, then it is said that an element $\mathbf{x}$ *majorized by* $\mathbf{y}$ and denoted $\mathbf{x} \prec \mathbf{y}$ if $\mathbf{x}_{[\downarrow]} \leq^b \mathbf{y}_{[\downarrow]}$. We refer readers to Marshall et al. (1979) for more information regarding this topic. One sees that b-order does not require a rearrangement of $\mathbf{x}$ by nonincreasing.

Any operator V with $V : S^{n-1} \to S^{n-1}$ is called *stochastic*. We call that a stochastic operator V is *b-bistochastic* if $V(\mathbf{x}) \leq^b \mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$. In what follows, we will study quadratic b-bistochastic operators.

Recall that $V : S^{n-1} \to S^{n-1}$ is called *quadratic stochastic operator (q.s.o.)* if V has a form

$$V : x_k^{'} = \sum_{i,j=1}^{n} P_{ij,k} x_i x_j \quad , k = 1, 2, ..., n \ , \tag{2.3}$$

where $\{P_{ij,k}\}$ are the heredity coefficients with the following properties:

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{n} P_{ij,k} = 1, \quad i, j, k = 1, 2, ..., n \ . \tag{2.4}$$

Remark 2.1. In Ganikhodzhaev (1993) q.s.o was introduced and studied with $V(x) \prec x$ for all $x \in S^{n-1}$. Such an operator is called *bistochastic*. In our definition we are taking b-order instead of the majorization. Note that if one takes absolute continuity instead of the b-order, then b-bistochastic operator reduces to Volterra q.s.o. (Mukhamedov (2000), Mukhamedov et al. (2016), Volterra and Brelot)

Theorem 2.2. *Mukhamedov and Embong (2015) Let $V : S^1 \to S^1$ be a q.s.o, then V is a $b-$bistochastic if and only if*

$$P_{22,1} = 0, \quad P_{12,1} \leq \frac{1}{2}.$$

Now let us consider a q.s.o. V defined on S^2. For the sake of simplicity let us denote

$$\begin{array}{lll} P_{11,1} = A_1 & P_{13,1} = C_1 & P_{23,1} = E_1 \\ P_{11,2} = A_2 & P_{13,2} = C_2 & P_{23,2} = E_2 \\ P_{12,1} = B_1 & P_{22,1} = D_1 & P_{33,1} = F_1 \\ P_{12,2} = B_2 & P_{22,2} = D_2 & P_{33,2} = F_2 \end{array} \tag{2.5}$$

and

$$M = 1-2C_1-2C_2,\ N = D_2-2E_2,\ P = 1-2E_2,\ Q = B_1+B_2-C_1-C_2-E_2,$$

$$R = (A_1 + A_2 - 2C_1 - 2C_2),\ K = 2(RN - Q^2)$$

$$a = A_1 + A_2 + D_2 - 2B_1 - 2B_2,\ b = 2B_1 + 2B_2 - 2D_2,\ c = D_2 - 1.$$

In Mukhamedov and Embong (2015) we have proved the following result.

Theorem 2.3. *Let $V : S^2 \to S^2$ be a q.s.o., then V is a b-bistochastic if and only if*

(i) $F_1 = E_1 = D_1 = F_2 = 0$

(ii) $B_1 \leq \frac{1}{2},\quad C_1 \leq \frac{1}{2},\quad E_2 \leq \frac{1}{2}$

(iii) $C_1 + C_2 \leq \frac{1}{2}$

and one of the followings are satisfied:

(I) $a > 0$

(II) $a < 0$ *and one of the followings are satisfied:*

- *(1)* $b < 0$
- *(2)* $b > -2a$
- *(3)* $b^2 - 4ac \leq 0$

Let T be a linear stochastic operator $T : S^{n-1} \to S^{n-1}$ such that

$$T(\mathbf{x})_k = \sum_{i=1}^{n} t_{ik}x_i \quad where \quad t_{ik} \geq 0,\ \sum_{k=1}^{n} t_{ik} = 1. \tag{2.6}$$

In fact, for every linear stochastic operator, it can be associated with a stochastic matrix. Define a q.s.o. V_T generated by heredity coefficient

$$P_{ij,k}^{(T)} = \frac{1}{2}\left(t_{ik} + t_{jk}\right),$$

then one gets $V_T(\mathbf{x}) = T(\mathbf{x})$. Therefore, V_T is b-bistochastic if and only if T is b-bistochastic.

Let T be a linear stochastic operator defined on two dimensional simplex and $\mathbb{T}$ be its corresponding matrix. It is clear that

$$\mathbb{T} = \begin{bmatrix} T(e_1) \\ T(e_2) \\ T(e_3) \end{bmatrix} \quad \textit{and} \quad \mathbf{x}\mathbb{T} = T(\mathbf{x}), \quad \mathbf{x} \in S^2 \tag{2.7}$$

From Theorem 2.3, one has the following result.

Corollary 2.4. *(Parker and Ram (1996)) Let T be a stochastic operator defined on S^2. Then, T is a b-bistochastic operator if and only if $\mathbb{T}$ is an upper triangular stochastic matrix, i.e,*

$$\mathbb{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Proof. Let T be b-bistochastic, then V_T is a b-bistochastic operator. Hence, from Theorem 2.3 we get

$$P_{33,1} = P_{23,1} = P_{33,2} = 0.$$

Thus, we obtain the following:

$$P_{33,1} = \frac{t_{31} + t_{31}}{2} \implies t_{31} = 0, \qquad P_{23,1} = \frac{t_{21} + t_{31}}{2} \implies t_{21} = 0, t_{31} = 0,$$
$$P_{33,2} = \frac{t_{32} + t_{32}}{2} \implies t_{32} = 0.$$

Evidently, we obtain $\mathbb{T}$ upper triangular stochastic matrix. Vice versa, we may define an operator given by (2.6), thus

$$T(\mathbf{x})_1 = t_{11}x_1 \leq x_1,$$
$$T(\mathbf{x})_1 + T(\mathbf{x})_2 = t_{12}x_1 + t_{22}x_2 \leq x_1 + x_2.$$

This shows that T is b-bistochastic. □

Note that, the last corollary can be extended to any finite dimensional simplex, and it was proven in Mukhamedov and Embong (2017), Parker and Ram (1996). Now we want to prove an analogue of a theorem of Hardy et al. (1986).

Theorem 2.5. *(Parker and Ram (1996)) Let* $\mathbf{x}, \mathbf{y} \in S^2$*, then* $\mathbf{y} \leq^b \mathbf{x}$ *if and only if there exists a b-bistochastic operator* T *such that* $T(\mathbf{x}) = \mathbf{y}$.

Proof. Clearly, if we have $T(\mathbf{x}) = \mathbf{y}$ and T is a b-bistochastic operator, then $\mathbf{y} \leq^b \mathbf{x}$. Conversely, let $\mathbf{y} \leq^b \mathbf{x}$. Now we want to show the existence of a $b-$bistochastic operator T such that $T(\mathbf{x}) = \mathbf{y}$. Due to Corollary 2.4, we need to find a stochastic matrix

$$\mathbb{T} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{bmatrix},$$

which satisfies $\mathbf{x}\mathbb{T} = \mathbf{y}$.

This means, one needs to solve the followings system:

$$ax_1 = y_1 \tag{2.8}$$

$$bx_1 + dx_2 = y_2 \tag{2.9}$$

$$cx_1 + ex_2 + x_3 = y_3 \tag{2.10}$$

Since (2.10) depends on (2.8) and (2.9), so it is enough to study the first two ones. Lets us show the existence of a, b and d in the interval [0,1] under the conditions $y_1 \leq x_1$ and $y_1 + y_2 \leq x_1 + x_2$. We consider several cases as follows:

Case 1. Let $x_1 = 0$, which implies $y_1 = 0$, since $y_1 \leq x_1$. Thus for any $a \in [0, 1]$,(2.8) holds. In fact, we have $y_2 \leq x_2$, therefore from (2.9) one finds $d = \dfrac{y_2}{x_2}$ and $d \in [0, 1]$ for any $b \in [0, 1 - a]$. This shows the existence of a, b and d.

Case 2. Consider $x_2 = 0$. Due to the b-bistochasticity of T, we know $\dfrac{y_1}{x_1} \leq 1$, therefore $a \in [0, 1]$ (see (2.8)). In this case, one finds that $y_2 \leq x_1$ which yields $b \leq 1$ (see (2.9)) for any $d \in [0, 1]$. Hence, a, b and d are found.

Case 3. In addition, we let $x_1 \neq 0$ and $x_2 \neq 0$. From (2.8) and the b-bistochasticity, one immediately gets $a \leq 1$. So, we found a and $0 \leq b \leq 1 - a$. Obviously, (2.9) can be simplified to

$$d = (-\frac{x_1}{x_2})b + \frac{y_2}{x_2}.$$

Denoting $f(b) = (-\dfrac{x_1}{x_2})b + \dfrac{y_2}{x_2}$, then the last equality means that $f(b) = d$. Let

$$d^* = \frac{y_2}{x_2}, \quad b^* = \frac{y_2}{x_1}, \quad f^{-1}(1) = \frac{y_2 - x_2}{x_1}.$$

Define

$$A = \{(b_0, d_0)|f(b_0) = d_0,\ 0 \leq b_0 \leq 1-a,\ 0 \leq d_0 \leq 1\}.$$

Further, we divide into four sub-cases, which are:

(i) $d^* \leq 1$ and $1-a \geq b^*$.

(ii) $d^* \leq 1$ and $1-a \leq d^*$.

(iii) $d^* \geq 1$ and $1-a \geq b^*$.

(iv) $d^* \geq 1$ and $1-a \leq d^*$.

Let us show for each sub-case, A is not empty.

(i) One can see that the set A is not empty since at least we have $(b^*, 0) \in A$. In fact, we have $f(b) = d \leq 1$ for all $b \in [0, b^*]$ which will be elements of set A.

(ii) In this case, we find for all $b \in [0, 1-a]$, $d \leq 1$ one has $(b, d) \in A$, so A is not empty.

(iii) One can check that $1-a \geq f^{-1}(1)$ holds, otherwise we have the following:

$$1 - \frac{y_1}{x_1} \leq \frac{y_2 - x_2}{x_1} \Longrightarrow x_1 + x_2 \leq y_1 + y_2,$$

which is a contradiction. Consequently, there exists an interval $I = [f^{-1}(1), 1-a]$ such that $f(b) = d \leq 1$ for all $b \in I$. This shows A is not empty.

(iv) The proof for this case follows from (i).

The proof is completed. $\square$

Next, let V be a b-bistochastic q.s.o. defined on S^2. Due to Theorem 2.5, there exists an upper triangular stochastic matrix $\mathbb{T}_{\mathbf{x}}$ such that $\mathbf{x}\mathbb{T}_{\mathbf{x}} = V(\mathbf{x})$. Let us assume $\mathbb{T}_{\mathbf{x}}$ depends on $\mathbf{x}$ linearly, i.e.

$$\mathbb{T}_{\lambda\mathbf{x}+\mu\mathbf{y}} = \lambda\mathbb{T}_{\mathbf{x}} + \mu\mathbb{T}_{\mathbf{y}}, \quad \mu, \lambda \geq 0,\ \mu + \lambda = 1. \tag{2.11}$$

So, this implies that, $\mathbb{T}_{\mathbf{x}}$ can be written as follows

$$\mathbb{T}_{\mathbf{x}} = x_1\mathbb{T}_{e_1} + x_2\mathbb{T}_{e_2} + x_3\mathbb{T}_{e_3}, \quad \mathbf{x} = (x_1, x_2, x_3) \in S^2.$$

Therefore,

$$V(\mathbf{x}) = \mathbf{x}\mathbb{T}_{\mathbf{x}} = x_1(\mathbf{x}\mathbb{T}_{e_1}) + x_2(\mathbf{x}\mathbb{T}_{e_2}) + x_3(\mathbf{x}\mathbb{T}_{e_3}), \quad \mathbf{x} \in S^2,$$

or

$$V(\mathbf{x})_k = \sum_{i,j=1}^{3} t_{ij,k} x_i x_j, \quad k = \{1, 2\} \tag{2.12}$$

The corresponding matrices $\mathbb{T}_{e_1}, \mathbb{T}_{e_2}$ and $\mathbb{T}_{e_3}$ are given by upper triangular stochastic matrices (see Corollary 2.4)

$$\begin{aligned}
\mathbb{T}_{e_1} &= \begin{bmatrix} t_{11,1} & t_{11,2} & t_{11,3} \\ 0 & t_{21,2} & t_{21,3} \\ 0 & 0 & 1 \end{bmatrix}, \mathbb{T}_{e_2} = \begin{bmatrix} t_{12,1} & t_{12,2} & t_{12,3} \\ 0 & t_{22,2} & t_{22,3} \\ 0 & 0 & 1 \end{bmatrix}, \\
\mathbb{T}_{e_3} &= \begin{bmatrix} t_{13,1} & t_{13,2} & t_{13,3} \\ 0 & t_{23,2} & t_{23,3} \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

Therefore, we are going to describe extreme b-bistochastic q.s.o. given by (2.12) in next section.

Remark 2.6. We have to stress that in general, $\mathbb{T}_{\mathbf{x}}$ may depend on $\mathbf{x}$ non-linearly.

In what follows, the operator given by (2.12) will be denoted by

$$V = (T_{e_1}, T_{e_2}, T_{e_3}) \tag{2.13}$$

Due to $P_{ij,k} = P_{ji,k}$ one finds

$$\begin{aligned}
V &= \left\{ \begin{bmatrix} P_{11,1} & P_{11,2} & P_{11,3} \\ 0 & P_{21,2} & P_{21,3} \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & P_{12,2} & P_{12,3} \\ 0 & P_{22,2} & P_{22,3} \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \right. \\
&\quad \left. ; \ P_{12,2} = P_{21,2} \ ; \sum_{k=1}^{3} P_{ij,k} = 1 \right\}.
\end{aligned} \tag{2.14}$$

3. Extreme Point Set of b-bistochastic QSO

If we consider the set of bistochastic linear operators, then it is important to describe extremal points of this set. In this setting, the description is given in Marshall et al. (1979). When, we look at the set of bistochastic q.s.o., then the description of extremal points is given in Ganikhodzhaev et al. (2012), Ganikhodzhaev (1993). Therefore, in the present paper, we are going to investigate the set of b-bistochastic q.s.o. and its extreme points as well.

Let us briefly recall some necessary notations. By $\mathcal{V}_b$ we denote the set of all b-bistochastic q.s.o. An element of $V \in \mathcal{V}_b$ is called extremal, if from $2V = V_1 + V_2$, where $V_1, V_2 \in \mathcal{V}_b$ it follows that $V = V_1 = V_2$. The set of all extremal points of $\mathcal{V}_b$ is denoted by $extr\mathcal{V}_b$. Here and henceforth $z = x \vee y$ means that z is either x or y. In Mukhamedov and Embong (2015), it was proved that $\mathcal{V}_b$ is a convex set. Thus, it is interesting to describe extreme points of the set $\mathcal{V}_b$. Due to the complexity to describe all extreme points in general cases, we restrict ourselves to low dimensional simplices.

3.1 Extremity on 1-D Simplex

In this subsection, we are going to describe extremal points of the set of b-bistochastic q.s.o. defined on one dimensional simplex. Note that each such kind of q.s.o. has the following form (see Theorem 2.2):

$$\left\{ \begin{bmatrix} P_{11,1} & P_{11,2} \\ P_{21,1} & P_{21,2} \end{bmatrix}, \begin{bmatrix} P_{12,1} & P_{12,2} \\ 0 & 1 \end{bmatrix} : \ P_{12,1} = P_{21,1} \leq \frac{1}{2}, \sum_{k=1}^{2} P_{ij,k} = 1 \right\}. \quad (3.1)$$

Theorem 3.1. *Let V be a b-bistochastic q.s.o. defined on one dimensional simplex. Then $V \in extr\mathcal{V}_b$ if and only if V has one of the following forms:*

$$(i) \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\} \quad (ii) \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

$$(iii) \left\{ \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \right\} \quad (iv) \left\{ \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \right\}$$

Proof. We will prove (i) is extremal point since the other three can be proven by the similar way. Let V_1 , $V_2 \in \mathcal{V}_b$, then one has

$$V_1 = \left\{ \begin{bmatrix} P^{(1)}_{11,1} & P^{(1)}_{11,2} \\ P^{(1)}_{21,1} & P^{(1)}_{21,2} \end{bmatrix}, \begin{bmatrix} P^{(1)}_{12,1} & P^{(1)}_{12,2} \\ 0 & 1 \end{bmatrix} \right\},$$

$$V_2 = \left\{ \begin{bmatrix} P^{(2)}_{11,1} & P^{(2)}_{11,2} \\ P^{(2)}_{21,1} & P^{(2)}_{21,2} \end{bmatrix}, \begin{bmatrix} P^{(2)}_{12,1} & P^{(2)}_{12,2} \\ 0 & 1 \end{bmatrix} \right\}.$$

Assume that

$$2V = V_1 + V_2, \quad (3.2)$$

where V is given by (i).

Obviously, (3.2) can be rewritten as follows

$$0 = \left(P^{(1)}_{11,1} - 2P^{(1)}_{12,1} + P^{(2)}_{11,1} - 2P^{(2)}_{12,1} - 2\right) x_1 + \left(2P^{(2)}_{12,1} + 2P^{(1)}_{12,1}\right).$$

Consequently, one gets

$$P^{(2)}_{12,1} + P^{(1)}_{12,1} = 0, \quad P^{(1)}_{11,1} + P^{(2)}_{11,1} = 2$$

The positivity and stochasticity of $P_{ij,k}$ imply

$$P^{(1)}_{12,1} = 0 \;\; P^{(2)}_{12,1} = 0 \;\; P^{(1)}_{11,1} = 1 \;\; P^{(2)}_{11,1} = 1,$$

which means

$$V_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\} \quad V_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\} \tag{3.3}$$

This shows that (i) is an extremal point of $\mathcal{V}_b$.

Conversely, if $V \in extr\mathcal{V}_b$, then from Theorem 2.2 one finds that the extremity of a b-bistochastic q.s.o. yields

$$P_{12,1} = 0 \vee \frac{1}{2}, \quad P_{11,1} = 0 \vee 1.$$

Thus, the only possible entries are given by (i),...,(iv). This completes the proof. □

3.2 Extremity on 2-D Simplex

By $\mathcal{V}_{0,1}$ we denote the set of q.s.o. given by (2.14) such that $P_{ij,k}$ may take 0 or 1 only.

Theorem 3.2. *One has* $\mathcal{V}_{0,1} \subset extr\mathcal{V}_b$

Proof. Let us show that for all elements in $\mathcal{V}_{0,1}$ belong to $extr\mathcal{V}_b$. Without loss of generality, we may choose the following matrix:

$$V = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}. \tag{3.4}$$

Assume that

$$2V = V_1 + V_2, \tag{3.5}$$

where $V_1, V_2 \in \mathcal{V}_b$ with

$$V_1 = \left\{ \begin{bmatrix} A_{1,1,1} & A_{1,1,2} & A_{1,1,3} \\ A_{2,1,1} & A_{2,1,2} & A_{2,1,3} \\ A_{3,1,1} & A_{3,1,2} & A_{3,1,3} \end{bmatrix}, \begin{bmatrix} A_{1,2,1} & A_{1,2,2} & A_{1,2,3} \\ 0 & A_{2,2,2} & A_{2,2,3} \\ 0 & A_{3,2,2} & A_{3,2,3} \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} A_{1,3,1} & A_{1,3,2} & A_{1,3,3} \\ 0 & A_{2,3,2} & A_{2,2,3} \\ 0 & 0 & 1 \end{bmatrix} \right\},$$
$$V_2 = \left\{ \begin{bmatrix} B_{1,1,1} & B_{1,1,2} & B_{1,1,3} \\ B_{2,1,1} & B_{2,1,2} & B_{2,1,3} \\ B_{3,1,1} & B_{3,1,2} & B_{3,1,3} \end{bmatrix}, \begin{bmatrix} B_{1,2,1} & B_{1,2,2} & B_{1,2,3} \\ 0 & B_{2,2,2} & B_{2,2,3} \\ 0 & B_{3,2,2} & B_{3,2,3} \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} B_{1,3,1} & B_{1,3,2} & B_{1,3,3} \\ 0 & B_{2,3,2} & B_{2,2,3} \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Denote $f(x_1, x_2) = V_1(\mathbf{x})_1 + V_2(\mathbf{x})_1 - 2V(\mathbf{x})_1$ and $g(x_1, x_2) = V_1(\mathbf{x})_2 + V_2(\mathbf{x})_2 - 2V(\mathbf{x})_2$, then one gets

$$\begin{aligned} f(x_1, x_2) &= (-A_{3,1,1} - A_{1,3,1} - B_{1,3,1} + B_{1,1,1} + A_{1,1,1} - B_{3,1,1} - 2)\, {x_1}^2 + \\ &\quad (A_{1,2,1} - A_{1,3,1} + A_{2,1,1} + B_{1,2,1} - A_{3,1,1} + B_{2,1,1} - B_{3,1,1} \\ &\quad - B_{1,3,1})\, x_2 x_1 + (B_{3,1,1} + A_{1,3,1} + A_{3,1,1} + B_{1,3,1})\, x_1 \end{aligned}$$

and

$$\begin{aligned} g(x_1, x_2) &= (A_{1,1,2} - B_{3,1,2} - A_{3,1,2} - B_{1,3,2} - A_{1,3,2} + B_{1,1,2})\, {x_1}^2 + \\ &\quad (B_{1,2,2} - A_{2,3,2} - A_{3,1,2} + A_{1,2,2} - A_{3,2,2} - A_{1,3,2} - B_{2,3,2} + \\ &\quad A_{2,1,2} - B_{3,1,2} + B_{2,1,2} - B_{1,3,2} - B_{3,2,2} - 4)\, x_2 x_1 + \\ &\quad (B_{1,3,2} + A_{3,1,2} + B_{3,1,2} + A_{1,3,2})\, x_1 + \\ &\quad (B_{2,2,2} - 2 - B_{3,2,2} - A_{2,3,2} - A_{3,2,2} + A_{2,2,2} - B_{2,3,2})\, {x_2}^2 + \\ &\quad (B_{3,2,2} + A_{2,3,2} + A_{3,2,2} + B_{2,3,2})\, x_2. \end{aligned}$$

From (3.5) we know that $f(x_1, x_2) = 0$ and $g(x_1, x_2) = 0$ for all $0 \leq x_1 + x_2 \leq 1$.

First, we investigate $f(x_1, x_2)$. One can see that

$$\begin{aligned} f(x_1,0) &= (-A_{3,1,1} - A_{1,3,1} - B_{1,3,1} + B_{1,1,1} + A_{1,1,1} - B_{3,1,1} - 2)\, {x_1}^2 + \\ &\quad (B_{3,1,1} + A_{1,3,1} + A_{3,1,1} + B_{1,3,1})\, x_1 = 0. \end{aligned}$$

By the same argument in the previous subsection, we get

$$B_{3,1,1} = A_{1,3,1} = A_{3,1,1} = B_{1,3,1} = 0.$$

Consequently, by the last equalities one gets

$$A_{1,1,1} = B_{1,1,1} = 1.$$

Hence, $f(x_1, x_2) = 0$ reduces to

$$(A_{1,2,1} + A_{2,1,1} + B_{1,2,1} + B_{2,1,1})\, x_2 x_1 = 0,$$

which means

$$A_{1,2,1} = A_{2,1,1} = B_{1,2,1} = B_{2,1,1} = 0.$$

Taking into account $g(x_1, x_2) = 0$, and letting $x_1 = 0$ (respectively $x_2 = 0$), then we have

$$\begin{aligned} 0 &= (B_{2,2,2} - 2 - B_{3,2,2} - A_{2,3,2} - A_{3,2,2} + A_{2,2,2} - B_{2,3,2})\, {x_2}^2 + \\ &\quad (B_{3,2,2} + A_{2,3,2} + A_{3,2,2} + B_{2,3,2})\, x_2. \end{aligned}$$

$$\begin{aligned} (\textit{respectively}\ \ 0 &= (A_{1,1,2} - B_{3,1,2} - A_{3,1,2} - B_{1,3,2} - A_{1,3,2} + B_{1,1,2})\, {x_1}^2 + \\ &\quad (B_{1,3,2} + A_{3,1,2} + B_{3,1,2} + A_{1,3,2})\, x_1.) \end{aligned}$$

Due to the stochasticity of the matrixes we find

$$B_{3,2,2} = A_{2,3,2} = A_{3,2,2} = B_{2,3,2} = 0, \quad A_{2,2,2} = B_{2,2,2} = 1,$$

$$B_{1,3,2} = A_{3,1,2} = B_{3,1,2} = A_{1,3,2} = 0, \quad A_{1,1,2} = B_{1,1,2} = 0.$$

Clearly, $g(x_1, x_2) = 0$ reduces to

$$(A_{1,2,2} + A_{2,1,2} + B_{1,2,2} + B_{2,1,2} + 4) x_1 x_2 = 0,$$

which means

$$A_{1,2,2} = A_{2,1,2} = B_{1,2,2} = B_{2,1,2} = 1.$$

The results show us

$$V_1 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

and

$$V_2 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

For other possibilities, using the same argument we obtain desired statements. This completes the proof. □

4. Quasi-Extreme of the set b-bistochastic q.s.o.

In Theorem 2.3 we showed that the set of all b-bistochastic q.s.o. defined on S^2 can be described by an 8-dimensional body (in terms of the heredity coefficients). Moreover, one concludes that the extreme points of $\mathcal{V}_b$ are descried by the extreme points of that body. Hence, to fully describe elements of $extr\mathcal{V}_b$ we need to find all elements of the 8-dimensional body, which is a tricky job. Instead of that, we would like to introduce a weaker condition than the extremity, called *quasi-extremity* which is less complicated. In what follows, we are representing $\{P_{ij,k}\}$ as in (2.5).

Definition 4.1. A b-bistochastic q.s.o. V defined on S^2 is called *quasi-exrtremal* if its corresponding heredity coefficients satisfy

(i) $B_1,\ C_1,\ E_2 = 0 \vee \frac{1}{2}$

(ii) $C_1 + C_2 = 0 \vee \frac{1}{2}$

Theorem 4.2. *Let T and S be linear b-bistochastic operators on S^{n-1}, then for any b-bistochastic quadratic operators V, the operator $S \circ V \circ T$ is also b-bistochastic quadratic operator.*

Proof. By using b-bistochasticity properties, we have ($\mathbf{x} = (x_1, \ldots, x_n) \in S^{n-1}$)

$$\sum_{i=1}^{k} S(V(T(\mathbf{x})))_i \leq \sum_{i=1}^{k} V(T(\mathbf{x}))_i \leq \sum_{i=1}^{k} T(\mathbf{x})_i \leq \sum_{i=1}^{k} x_i, \quad \text{for all } k = \overline{1, n-1}$$

this completes the proof. □

Theorem 4.3. *Let $T = (T_{ij})$ and $S = (S_{ij})$ be linear stochastic operators on S^{n-1} and let V be a q.s.o. on S^{n-1}. We denote*

$$q_{uv,k} = \left(\sum_{i,j,l=1}^{n} P_{ij,l} \cdot T_{u,i} \cdot T_{v,j} \cdot S_{l,k} \right).$$

If $q_{uv,k} = q_{vu,k}$*, then* $S \circ V \circ T$ *is a q.s.o.*

Proof. For any $k = \{1, 2, \ldots, n\}$ one sees that

$$\begin{aligned}
S(V(T(\mathbf{x})))_k &= \sum_{l=1}^{n} S_{l,k}(V(T(\mathbf{x}))_l) \\
&= \sum_{l=1}^{n} S_{l,k}\left(\sum_{i,j=1}^{n} P_{ij,l}\left(T(\mathbf{x})_i T(\mathbf{x})_j\right)\right) \\
&= \sum_{l=1}^{n} S_{l,k}\left(\sum_{i,j=1}^{n} P_{ij,l}\left(\sum_{u=1}^{n} T_{u,i}x_u\right)\left(\sum_{v=1}^{n} T_{v,j}x_v\right)\right) \\
&= \sum_{l=1}^{n}\sum_{i,j=1}^{n}\sum_{u=1}^{n}\sum_{v=1}^{n}\left(S_{l,k}\cdot P_{ij,l}\cdot T_{u,i}\cdot T_{v,j}\right)x_u x_v \\
&= \sum_{u,v=1}^{n}\left(\sum_{i,j,l=1}^{n} P_{ij,l}\cdot T_{u,i}\cdot T_{v,j}\cdot S_{l,k}\right)x_u x_v.
\end{aligned}$$

We may write

$$S(V(T(\mathbf{x})))_k = \sum_{u,v=1}^{n} q_{uv,k}x_u x_v$$

where $q_{uv,k} = \left(\sum_{i,j,l=1}^{n} P_{ij,l}\cdot T_{u,i}\cdot T_{v,j}\cdot S_{l,k}\right)$. It is clear that $\sum_{k=1}^{n} S(V(T(\mathbf{x})))_k = 1$

By assumption, one infer the operator $S(V(T(\mathbf{x})))$ is QSO. The proof is completed. $\square$

From Theorem 4.2, one naturally arises the following problem: if S and T are extreme b-bistochastic linear operators, then how b-bistochastic q.s.o. V and operator $S \circ V \circ T$ are related to each other, in terms of quasi-extremity.

First, we denote all possible extreme points of b-bistochastic linear stochastic operators, which can be described by

$$L_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad L_2=\begin{bmatrix}0&1&0\\0&1&0\\0&0&1\end{bmatrix},\quad L_3=\begin{bmatrix}0&0&1\\0&1&0\\0&0&1\end{bmatrix},$$
$$L_4=\begin{bmatrix}1&0&0\\0&0&1\\0&0&1\end{bmatrix},\quad L_5=\begin{bmatrix}0&1&0\\0&0&1\\0&0&1\end{bmatrix},\quad L_6=\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}.$$

Due to (2.14) any b-bistochastic q.s.o. has the following form

$$V = \left\{\begin{bmatrix}A_1&A_2&1-A_1-A_2\\B_1&B_2&1-B_1-B_2\\C_1&c_2&1-C_1-C_2\end{bmatrix},\begin{bmatrix}B_1&B_2&1-B_1-B_2\\0&D_2&1-D_2-D_2\\0&E_2&1-D_1-D_2\end{bmatrix},\right.$$
$$\left.\begin{bmatrix}C_1&C_2&1-C_1-C_2\\0&E_2&1-D_1-D_2\\0&0&1\end{bmatrix}\right\}$$

Henceforth, by $Q_{a,b}$ we denote the operator $L_a\circ V\circ L_b$ $(a,b=\overline{1,6})$ and its cubic matrix we denote by $\{q_{e_1},q_{e_2},q_{e_3}\}$, where $q_{e_v}=(q_{uv,k})_{u,k=1}^3$. Here

$$q_{uv,k}=\sum_{i,j,l=1}^{3}P_{ij,l}\cdot L_b(u,i)\cdot L_b(v,j)\cdot L_a(l,k).$$

where $\{P_{ij,k}\}$ are hereditary coefficients of V, whereas $L_c(i,j)$ is the element of the matrix L_c $(c=\overline{1,6})$ at i^{th} row and j^{th} column. Using Maple, we find

$$Q_{1,2}=Q_{2,2},\quad Q_{1,3}=Q_{2,3},\quad Q_{1,5}=Q_{2,5}=Q_{3,5}, \tag{4.1}$$
$$Q_{1,6}=Q_{2,6}=Q_{3,6}=Q_{4,2}=Q_{4,3}=Q_{4,5}=Q_{4,6}=Q_{5,2}=$$
$$Q_{5,3}=Q_{5,5}=Q_{5,6}=Q_{6,1}=Q_{6,2}=Q_{6,3}=Q_{6,4}=Q_{6,5}=Q_{6,6} \tag{4.2}$$

All possible operators can be described by

$$Q_{1,1}=\left\{\begin{bmatrix}A_1&A_2&1-A_1-A_2\\B_1&B_2&1-B_1-B_2\\C_1&C_2&1-C_1-C_2\end{bmatrix},\begin{bmatrix}B_1&B_2&1-B_1-B_2\\0&D_2&1-D_2\\0&E_2&1-E_2\end{bmatrix}.\begin{bmatrix}C_1&C_2&1-C_1-C_2\\0&E_2&1-E_2\\0&0&1\end{bmatrix}\right\}$$
$$Q_{1,2}=\left\{\begin{bmatrix}0&D_2&1-D_2\\0&D_2&1-D_2\\0&E_2&1-E_2\end{bmatrix},\begin{bmatrix}0&D_2&1-D_2\\0&D_2&1-D_2\\0&E_2&1-E_2\end{bmatrix},\begin{bmatrix}0&E_2&1-E_2\\0&E_2&1-E_2\\0&0&1\end{bmatrix}\right\}$$

$$Q_{1,3}=\left\{\begin{bmatrix}0&0&1\\0&E_2&1-E_2\\0&0&1\end{bmatrix},\begin{bmatrix}0&E_2&1-E_2\\0&D_2&1-D_2\\0&E_2&1-E_2\end{bmatrix},\begin{bmatrix}0&0&1\\0&E_2&1-E_2\\0&0&1\end{bmatrix}\right\}$$

$$Q_{1,4}=\left\{\begin{bmatrix}A_1&A_2&1-A_1-A_2\\C_1&C_2&1-C_1-C_2\\C_1&C_2&1-C_1-C_2\end{bmatrix},\begin{bmatrix}C_1&C_2&1-C_1-C_2\\0&0&1\\0&0&1\end{bmatrix},\begin{bmatrix}C_1&C_2&1-C_1-C_2\\0&0&1\\0&0&1\end{bmatrix}\right\}$$

$$Q_{1,5}=\left\{\begin{bmatrix}0&D_2&1-D_2\\0&E_2&1-E_2\\0&E_2&1-E_2\end{bmatrix},\begin{bmatrix}0&E_2&1-E_2\\0&0&1\\0&0&1\end{bmatrix},\begin{bmatrix}0&E_2&1-E_2\\0&0&1\\0&0&1\end{bmatrix}\right\}$$

$$Q_{1,6}=\left\{\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\right\}$$

$$Q_{2,1}=\left\{\begin{bmatrix}0&A_1+A_2&1-A_1-A_2\\0&B_1+B_2&1-B_1-B_2\\0&C_1+C_2&1-C_1-C_2\end{bmatrix},\begin{bmatrix}0&B_1+B_2&1-B_1-B_2\\0&D_2&1-D_2\\0&E_2&1-E_2\end{bmatrix},\right.$$
$$\left.\begin{bmatrix}0&C_1+C_2&1-C_1-C_2\\0&E_2&1-E_2\\0&0&1\end{bmatrix}\right\}$$

$$Q_{2,4}=\left\{\begin{bmatrix}0&A_1+A_2&1-A_1-A_2\\0&C_1+C_2&1-C_1-C_2\\0&C_1+C_2&1-C_1-C_2\end{bmatrix},\begin{bmatrix}0&C_1+C_2&1-C_1-C_2\\0&0&1\\0&0&1\end{bmatrix},\right.$$
$$\left.\begin{bmatrix}0&C_1+C_2&1-C_1-C_2\\0&0&1\\0&0&1\end{bmatrix}\right\}$$

$$Q_{3,1}=\left\{\begin{bmatrix}0&A_2&1-A_2\\0&B_2&1-B_2\\0&C_2&1-C_2\end{bmatrix},\begin{bmatrix}0&B_2&1-B_2\\0&D_2&1-D_2\\0&E_2&1-E_2\end{bmatrix},\begin{bmatrix}0&C_2&1-C_2\\0&E_2&1-E_2\\0&0&1\end{bmatrix}\right\}$$

$$Q_{3,4}=\left\{\begin{bmatrix}0&A_2&1-A_2\\0&C_2&1-C_2\\0&C_2&1-C_2\end{bmatrix},\begin{bmatrix}0&C_2&1-C_2\\0&0&1\\0&0&1\end{bmatrix},\begin{bmatrix}0&C_2&1-C_2\\0&0&1\\0&0&1\end{bmatrix}\right\}$$

$$Q_{4,1}=\left\{\begin{bmatrix}A_1&0&1-A_1\\B_1&0&1-B_1\\C_1&0&1-C_1\end{bmatrix},\begin{bmatrix}B_1&0&1-B_1\\0&0&1\\0&0&1\end{bmatrix},\begin{bmatrix}C_1&0&1-C_1\\0&0&1\\0&0&1\end{bmatrix}\right\}$$

$$Q_{4,4}=\left\{\begin{bmatrix}A_1&0&1-A_1\\C_1&0&1-C_1\\C_1&0&1-C_1\end{bmatrix},\begin{bmatrix}C_1&0&1-C_1\\0&0&1\\0&0&1\end{bmatrix},\begin{bmatrix}C_1&0&1-C_1\\0&0&1\\0&0&1\end{bmatrix}\right\}$$

$$Q_{5,1}=\left\{\begin{bmatrix}0&A_1&1-A_1\\0&B_1&1-B_1\\0&C_1&1-C_1\end{bmatrix},\begin{bmatrix}0&B_1&1-B_1\\0&0&1\\0&0&1\end{bmatrix},\begin{bmatrix}0&C_1&1-C_1\\0&0&1\\0&0&1\end{bmatrix}\right\}$$

$$Q_{5,4}=\left\{\begin{bmatrix}0&A_1&1-A_1\\0&C_1&1-C_1\\0&C_1&1-C_1\end{bmatrix},\begin{bmatrix}0&C_1&1-C_1\\0&0&1\\0&0&1\end{bmatrix},\begin{bmatrix}0&C_1&1-C_1\\0&0&1\\0&0&1\end{bmatrix}\right\}$$

Theorem 4.4. *Let V be a b-bistochastic q.s.o. defined on S^2, then for any $a, b = \overline{1,6}$, $Q_{a,b}$ are b-bistochastic q.s.o.*

The proof directly follows from Theorem 4.2 and above listed forms.

The next theorem is the main result in this section that shows how operators $Q_{a,b}$ are related to V in terms of quasi-extremity.

Theorem 4.5. *Let $V : S^2 \to S^2$ be a b-bistochastic q.s.o., then the following statements hold:*

(i) V is quasi-extreme if and only if $Q_{1,1}$ is quasi-extreme;

(ii) the operators $Q_{a,b}$ given by (4.2) are quasi-extreme regardless of V;

(iii) if V is quasi-extreme, then the operators $Q_{a,b}$ (other than in (i) and (ii)) are quasi-extreme

Proof. (i) This proof is clear since $Q_{1,1} = V$.
(ii) Taking into account $Q_{1,6}$, then it is obvious $Q_{1,6}$ is quasi-extreme regardless of V.
(iii) Let us consider $Q_{1,2}$. If V is quasi-extreme, it immediately implies $E_2 = 0 \vee \frac{1}{2}$. This means $Q_{1,2}$ is quasi-extreme. Note that, the reverse may not be true since a quasi-extremity of $Q_{1,2}$ does not imply $B_1, C_1, E_2, C_1 + C_2 = 0 \vee \frac{1}{2}$. The other cases can be done by the same argument. □

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